LESSON

3 Exponents and Logarithms

Exponents

Because exponents form such an important part of calculus, we shall briefly review them. Generally, a^n means "multiply the base a as many times as the exponent n."

$$a^n = \underbrace{a \cdot a \cdot a \cdots a}_{n\ times}$$

Note: The exponent formulas in this lesson all assume that a is a positive number.

Examples

Review the following examples by multiplying out.

$3^4 = 3 \cdot 3 \cdot 3 \cdot 3 = 81$

$2^5 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 32$

$5^1 = 5$

$10^6 = 1{,}000{,}000$

A number to the first power is just that number:
$a^1 = a$

When two numbers with the same base are multiplied, their exponents are added.

$$a^n \cdot a^m = \underbrace{(a \cdot a \cdot a \cdots a)}_{n\ times} \cdot \underbrace{(a \cdot a \cdot a \cdots a)}_{m\ times} = a^{n+m}$$

Examples

Review and simplify the following.

$4^{10} \cdot 4^7 = 4^{17}$

$10^2 \cdot 10^5 = 10^7$

$5^3 \cdot 5 = 5^3 \cdot 5^1 = 5^4$

$7^2 \cdot 7^4 \cdot 7^3 = 7^9$

The rule about adding exponents has an interesting consequence. We know that $\sqrt{5} \cdot \sqrt{5} = 5$ because this is what "square root" means. Also, however, $5^{\frac{1}{2}} \cdot 5^{\frac{1}{2}} = 5^{\frac{1}{2}+\frac{1}{2}} = 5^1 = 5$. Because $\sqrt{5}$ and $5^{\frac{1}{2}}$ act exactly the same, they are equal: $\sqrt{5} = 5^{\frac{1}{2}}$. This works for square roots, cube roots, and so on:

$$a^{\frac{1}{2}} = \sqrt{a},\ a^{\frac{1}{3}} = \sqrt[3]{a},\ a^{\frac{1}{4}} = \sqrt[4]{a}, \ldots$$

Examples

Simplify the following.

$9^{\frac{1}{2}} = \sqrt{9} = 3$

$63^{\frac{1}{3}} = \sqrt[3]{64} = 4$

When two numbers with the same base are divided, their exponents are subtracted.

$$\frac{a^n}{a^m} = a^{n-m}$$

Examples

Work through the following simplifications.

$$\frac{3^5}{3^2} = \frac{3 \cdot 3 \cdot 3 \cdot 3 \cdot 3}{3 \cdot 3} = \frac{3 \cdot 3 \cdot 3 \cdot \cancel{3 \cdot 3}}{\cancel{3 \cdot 3}} = 3 \cdot 3 \cdot 3 = 3^3$$

$$\frac{11^{15}}{11^6} = 11^9$$

The rule about subtracting exponents has two interesting consequences. First, $\frac{5^4}{5^4} = 1$ because any nonzero number divided byitself is one. Also, $\frac{5^4}{5^4} = 5^{4-4} = 5^0$. Thus, $5^0 = 1$. In general:

$a^0 = 1$

Simplify the following.
$3^0 = 1$
$200^0 = 1$

The second consequence follows from:

$$\frac{2^3}{2^7} = \frac{\cancel{2 \cdot 2 \cdot 2}}{2 \cdot 2 \cdot 2 \cdot 2 \cdot \cancel{2 \cdot 2 \cdot 2}} = \frac{1}{2 \cdot 2 \cdot 2 \cdot 2} = \frac{1}{2^4}$$ while

also $\frac{2^3}{2^7} = 2^{3-7} = 2^{-4}$. Thus, $2^{-4} = \frac{1}{2^4}$. In general:

$$a^{-n} = \frac{1}{a^n}$$

Examples

Work through the following simplifications.

$$3^{-2} = \frac{1}{3^2} = \frac{1}{9}$$

$$4^{-1} = \frac{1}{4^1} = \frac{1}{4}$$

$$5^{-\frac{1}{2}} = \frac{1}{5^{\frac{1}{2}}} = \frac{1}{\sqrt{5}}$$

▶ Practice

Simplify the following.

1. $2^3 \cdot 2^2$

2. $4 \cdot 4^2$

3. $\frac{10^7}{10^3}$

4. $\frac{6^3}{6^5}$

5. 6^0

6. $3^8 \cdot 3 \cdot 3^{-5}$

7. 9^1

8. $25^{\frac{1}{2}}$

9. 5^{-1}

10. $8^{\frac{1}{3}}$

11. 2^{-3}

12. $8^{\frac{2}{3}}$

13. $\frac{1}{5^{-1}}$

14. 10^{-5}

15. $\frac{1}{8^{-2}}$

16. $\frac{1}{16^{-\frac{1}{2}}}$

▶ Exponential Functions

We can form an *exponential function* by leaving the base fixed and varying the exponent.

Example

The function $f(x) = 2^x$ has the graph shown in Figure 3.1. Note that 2^x is quite different from x^2. For example, when $x = 10$, the value of 2^x is $2^{10} = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 1{,}024$, while the value of x^2 is $10^2 = 10 \cdot 10 = 100$.

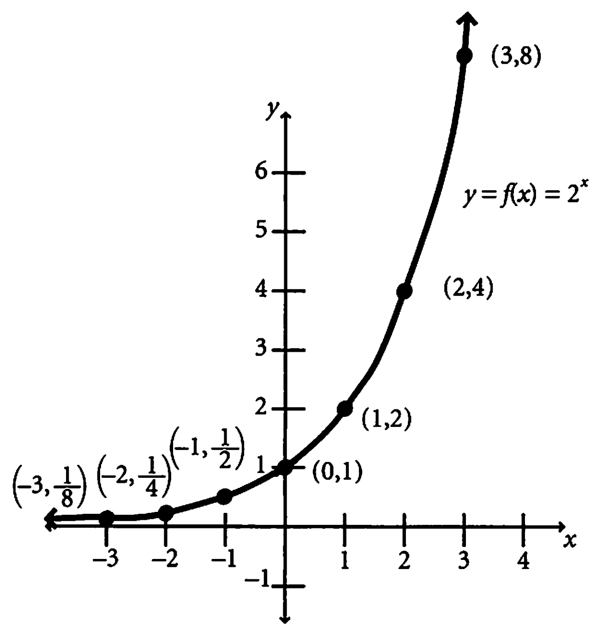

Figure 3.1

Example

The function $g(x) = 3^x$ has the graph shown in Figure 3.2. For reasons that will become clear later, a very nice base to use is the number $e = 2.71828\ldots$, which, just like $\pi = 3.14159\ldots$, can never be written out completely.

Exponents and Logarithms

The exponential function takes x to e^x and the natural logarithm takes it right back to x, so $\ln(e^x) = x$.

Similarly, $e^{\ln(x)} = x$.

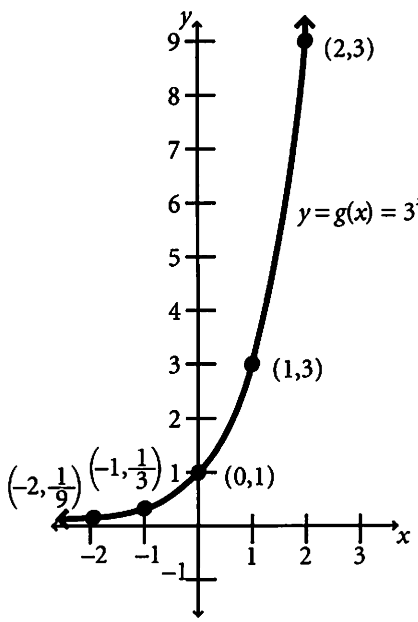

Figure 3.2

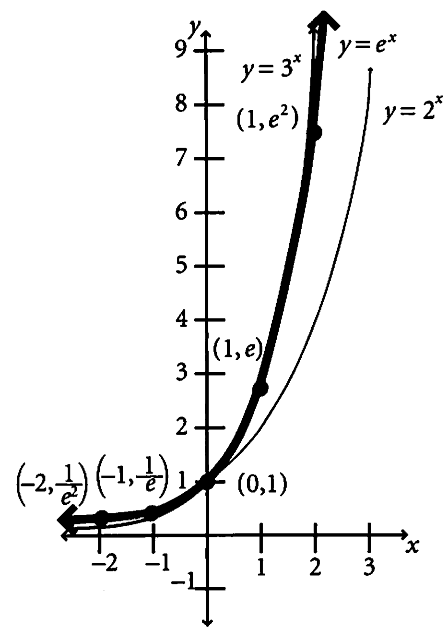

Figure 3.3

Because $2 < e < 3$, the graph of $y = e^x$ fits between $y = 2^x$ and $y = 3^x$ (see Figure 3.3).

Other than the strange base, everything about e^x is normal.

$e^0 = 1$

$e^n \cdot e^m = e^{n+m}$

$e^1 = e$

$\frac{e^n}{e^m} = e^{n-m}$

Another useful function is the opposite of e^x, known as the *natural logarithm* $\ln(x)$. Just as subtracting undoes adding, dividing undoes multiplying, and taking a square root undoes squaring, the natural logarithm undoes e^x.

If $y = e^x$, then $\ln(y) = \ln(e^x)$, so $\ln(y) = x$.

The graph of $y = \ln(x)$ comes from flipping the graph of $y = e^x$ across the line $y = x$, as depicted in Figure 3.4.

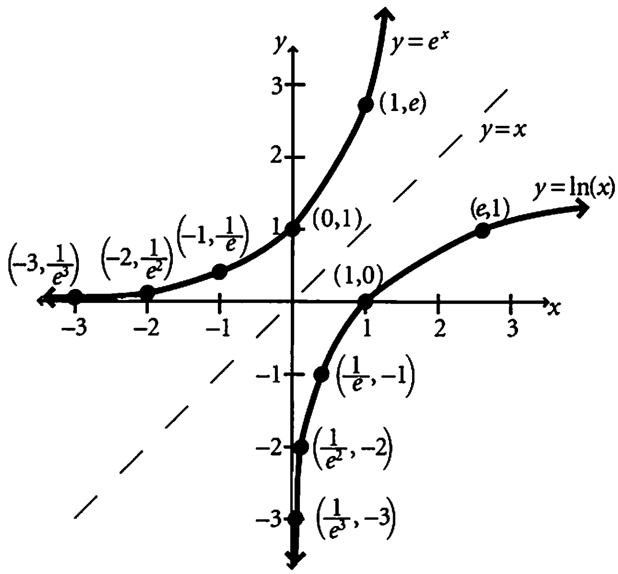

Figure 3.4

The laws of $\ln(x)$ are rather unusual.

$$\ln(a) + \ln(b) = \ln(a \cdot b)$$

$$\ln(a) - \ln(b) = \ln\left(\frac{a}{b}\right)$$

$$\ln(a^n) = n \cdot \ln(a)$$

The last of the three preceding laws is useful for turning an exponent into a matter of multiplication.

Example
Solve for x when $10^x = 7$.

Solution
Take the natural logarithm of both sides.

$$\ln(10^x) = \ln(7)$$

Use $\ln(a^n) = n \cdot \ln(a)$.

$$x \cdot \ln(10) = \ln(7)$$

Divide both sides by $\ln(10)$.

$$x = \frac{\ln(7)}{\ln(10)}$$

A calculator can be used to find a decimal approximation: $\frac{\ln(7)}{\ln(10)} \approx 0.84509$, if desired.

Example
Simplify $\ln(25) + \ln(4) - \ln(2)$.

Solution
Use $\ln(a) + \ln(b) = \ln(a \cdot b)$.

$$\ln(25 \cdot 4) - \ln(2)$$

Use $\ln(a) - \ln(b) = \ln\left(\frac{a}{b}\right)$.

$$\ln\left(\frac{25 \cdot 4}{2}\right) = \ln(50)$$

▶ Practice

Simplify the following.

17. $e^3 \cdot e^8$

18. $\frac{e^{12}}{e^5}$

19. e^0

20. $\ln(e^2)$

21. $e^{\ln(5)}$

22. $\ln(7) + \ln(2)$

23. $\ln(24) - \ln(6)$

24. Solve for x when $2^x = 10$.

25. Solve for x when $8^x = 11$.

26. Solve for x when $3^x \cdot 3^5 = 100$.